

# Approximate Analytic Spectra of Reionized CMB Anisotropies and Polarization generated by Relic Gravitational Waves

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## Abstract

We present an approximate, analytical calculation of the reionized spectra  $C_l^{XX}$  of cosmic microwave background radiation (CMB) anisotropies and polarizations generated by relic gravitational waves (RGWs). Three simple models of reionization are explored, whose visibility functions are fitted by gaussian type of functions as approximations. We have derived the analytical polarization  $\beta_l$  and temperature anisotropies  $\alpha_l$ , both consisting of two terms proportional to RGWs at the decoupling and at the reionization as well. The explicit dependence of  $\beta_l$  and  $\alpha_l$  upon the reionization time  $\eta_r$ , the duration  $\Delta\eta_r$ , and the optical depth  $\kappa_r$  are demonstrated. Moreover,  $\beta_l$  and  $\alpha_l$  contain  $\kappa_r$  in different coefficients, and the polarization spectra  $C_l^{EE}$  are  $C_l^{BB}$  are more sensitive probes of reionization than  $C_l^{TT}$ . These results facilitate examination of the reionization effects, in particular, the degeneracies of  $\kappa_r$  with the normalization amplitude and with the initial spectral index of RGWs.

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It is also found that reionization also causes a  $\kappa_r$ -dependent shift  $\Delta l \sim 20$  of the zero multipole  $l_0$  of  $C_l^{TE}$ , an effect that should be included in order to detect the traces of RGWs. Compared with numerical results, the analytical  $C_l^{XX}$  as approximation have the limitation. For the primary peaks in the range  $l \simeq (30, 600)$ , the error is  $\leq 3\%$  in three models. In the range  $l < 20$  for the reionization bumps, the error is  $\leq 15\%$  for  $C_l^{EE}$  and  $C_l^{BB}$  in the two extended reionization models, and  $C_l^{TT}$  and  $C_l^{TE}$  have much larger departures for  $l < 10$ . The bumps in the sudden reionization model are too low.

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# 1. Introduction

Reionization is a very important cosmological process, which might be, to a large extent, determined by the first luminous objects formed in the early universe, either star-forming galaxies or active galactic nuclei. Our knowledge of the cosmic structure formation of the universe would be incomplete without a reliable account of reionization history, the details of which is still not understood yet. During the evolution history of CMB, the reionization taking place around the redshift  $z = (20 \sim 6)$  is a major process in shaping the profiles of CMB spectra on large scales, only secondary to the decoupling around  $z \sim 1100$ . Reionization leaves observable prints on CMB [1, 2, 3, 4, 5, 6, 7, 8] through the interaction between the CMB photons and the reionized free electrons. In particular, the spectra of CMB anisotropies and polarizations on large angular scales contain the distinguished signatures of reionization. Thereby, complementary to the constraints on the late stage of reionization  $z \simeq 6$  from observations of the most distant quasars absorption lines, etc, CMB provides a unique probe for the early stage of reionization. On the other hand, in order to interpret the observed spectra of CMB anisotropies and polarizations within the standard model, the reionization-induced modifications have to be taken into account properly. As is known, the reionization parameters could be entangled with the cosmological parameters, thus biasing our interpretation of CMB, and of reionization as well [9, 10, 11, 12, 13, 14]. In this regards, analytic studies can improve our understanding of CMB and reionization, even though the comparisons with the observed data need more accurate numerical calculations, such as cmbfast and CAMB [15, 16].

Two kinds of perturbations of the spacetime metric, i.e., density perturbation [17, 18, 19] and relic gravitational waves (RGWs) [18, 20, 21, 22], will effectively influence the CMB through the Sachs-Wolfe term [23] in the Boltzmann equation for photons. Although the contribution by density perturbation is dominant, RGWs give rise to a magnetic type of CMB polarizations, providing a distinguished channel to directly detect RGWs of very long wavelength [9, 11, 26, 27]. Moreover, RGWs have substantial contributions to large angular scales part of CMB spectra, where the impact of reionization is also dominant and cause bumps in the CMB polarization spectra for  $l < 10$ . Thus, in order to study reionization through the CMB, one has to take into account of the contribution of RGWs, or, vice versa.

The analyses have been made towards CMB anisotropies and polarizations generated by RGWs [20, 21, 24, 25, 27, 28, 29, 30, 31]. In particular, by an approximate treatment of the time integration over the decoupling process during the recombination, Refs.[32, 33] have derived the analytic expressions of the CMB polarization spectra,  $C_l^{EE}$  and  $C_l^{BB}$ . Recently,

extending the previous works, we have improved the time integration by a better approximation, and obtained the analytical expressions of all the four spectra, including  $C_l^{TT}$  and  $C_l^{TE}$  [34], which agree fairly with the numerical results up to a broader range of multipole moment  $l < 600$ . In that work the damping on RGWs due to neutrino free-streaming (NFS) has been included [35, 36, 37, 38], and its effects on the cross spectrum  $C_l^{TE}$  have been demonstrated in details. In these analytical calculations, the reionization process has not been included, which will be addressed in this paper. For the purpose of calculating the reionized CMB spectra  $C_l^{XX}$ , the reionization can be treated similarly to the decoupling, if the visibility functions for both processes are given. While the decoupling and its visibility function  $V_d(\eta)$  effectively distributed around  $z \sim 1100$  have been better studied, the reionization is currently less understood, and is commonly modeled by its ionization fraction  $X_e(\eta)$  as a function of time. We shall examine three possible simple reionization models with explicit  $X_e(\eta)$ , which, for a given value of the optical depth  $\kappa_r$ , can be converted into its corresponding visibility function  $V_r(\eta)$  effectively distributed around  $z \sim 11$ . The functions  $V_d(\eta)$  and  $V_r(\eta)$  are separately distributed, not overlapping, each of them can respectively be approximated by Gaussian type of functions, which are specified by their location, height, and width. In parallel, we will carry out, with approximation, the time integrations of Boltzmann's equation for the decoupling and reionization processes. The modes  $\alpha_l$  and  $\beta_l$ , respectively, for CMB temperature anisotropies and polarization, are obtained as analytical expressions. Each mode explicitly consists of two separated parts, one from the decoupling, and another from the reionization. Moreover, the optical depth  $\kappa_r$  appears as the coefficients in  $\alpha_l$  and  $\beta_l$  in different combinations, and probabilistic interpretations are given. Besides reionization and decoupling, the result contains also other cosmological parameters for inflation that are contained in RGWs. Thus analytic studies on the reionization effects will be facilitated.

In Section 2 we review briefly the result of RGWs spectrum  $h(\nu, \eta)$  that will be used as the source for CMB anisotropies and polarization. In Section 3 three models of homogeneous reionization  $X_e(\eta)$  are presented: one sudden and two extended. For each model the visibility function  $V_r(\eta)$  and the optical depth function  $\kappa_r(\eta)$  are presented. In Section 4, by approximately carrying out the time integrations, the analytical expressions of  $\alpha_l$  and  $\beta_l$  are obtained. The resulting spectra  $C_l^{XX}$  are demonstrated. In Section 5, detailed analyses are made towards the reionization effects upon  $C_l^{XX}$ , three models are compared, and, in particular, examinations are made on the degeneracies of  $\kappa_r$  with the normalization amplitude  $A$  and the initial spectral index  $\beta_{inf}$  of RGWs produced during inflation. The effect of reionization on the zero multipole analysis is addressed. The conclusion is given in Section 6. We use the unit in which  $c = \hbar = k_B = 1$  in

this paper.

## 2. RGWs Spectrum

The expansion of a spatially flat Universe can be described by the spatially flat ( $\Omega_\Lambda + \Omega_m + \Omega_r = 1$ ) Robertson-Walker spacetime with a metric

$$ds^2 = a^2(\eta)[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \quad (1)$$

where  $a(\eta)$  is the scale factor,  $\eta$  is the conformal time, and  $h_{ij}$  is the gravitational waves, taken to be traceless and transverse (TT gauge)  $h^i_i = 0$ , and  $h_{ij,j} = 0$ . By the Fourier decomposition

$$h_{ij}(\eta, \mathbf{x}) = \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \epsilon_{ij}^{\sigma} h_{\mathbf{k}}^{(\sigma)}(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (2)$$

for each mode  $\mathbf{k}$  and each polarization  $\sigma = (+, \times)$ , the wave equation takes the form

$$\ddot{h}_k + 2\frac{\dot{a}}{a}\dot{h}_k + k^2 h_k = 0, \quad (3)$$

where the polarization index  $\sigma$  has been skipped for simplicity, and the subindex  $\mathbf{k}$  can be replace by  $k$  since the perturbations are assumed to be isotropic. The analytic solution of Eq.(3) has been given for the expanding universe with the consecutive stages: inflationary, reheating, radiation-dominant, matter-dominant, and accelerating, respectively in Refs.[38, 39, 40]. In our convention,

$$a(\eta) = a_m(\eta - \eta_m)^2, \quad \eta_2 \leq \eta \leq \eta_E, \quad (4)$$

for the matter-dominant stage, and

$$a(\eta) = l_H |\eta - \eta_a|^{-\gamma}, \quad \eta_E \leq \eta \leq \eta_0, \quad (5)$$

for accelerating stage up to the present time  $\eta_0$ , where  $\gamma \simeq 1.044$  for  $\Omega_\Lambda = 0.75$ , and  $l_H = \gamma/H_0$ ,  $H_0$  is the Hubble constant. The normalization of  $a(\eta)$  is chosen to be  $|\eta_0 - \eta_a| = \eta_a - \eta_0 = 1$ , where we have taken  $\eta_0 = 3.11$  to be the present time. Then, once the ratio  $\Omega_\Lambda/\Omega_m$  is specified, all the parameters will fixed:  $a_m = l_H \frac{\gamma^2}{4} \zeta_E^{-(1+2/\gamma)}$ ,  $\eta_E = \eta_a - \zeta_E^{1/\gamma}$ ,  $\eta_m = \eta_E - \frac{2}{\gamma} \zeta_E^{1/\gamma}$  with  $\zeta_E \equiv (\Omega_\Lambda/\Omega_m)^{1/3}$ . The details have been explicitly demonstrated in our previous study of RGWs [38, 39].

When the NFS is included, a process occurred from a temperature  $T \simeq 2$  MeV during the radiation stage up to the beginning of the matter domination, the analytic solution  $h_k(\eta)$  has been given [34, 38]. The NFS causes a damping of the amplitude of RGWs by  $\sim 20\%$  in the

frequency range ( $10^{-17}$ ,  $10^{-10}$ ) Hz, leaving observable signatures on the second and third peaks of CMB anisotropies and polarization. So the RGWs damped by NFS will be used as a source in our calculation. As for other physical processes, such as the QCD transition and the  $e^\pm$  annihilation in the radiation stage [37, 40, 41], they only cause minor modifications of RWGs on the small scales  $\nu > 10^{-12}$  Hz, not being observable in the present large-scale CMB spectra, and will not be considered here.

The solution  $h_k(\eta)$  depends on the initial condition during the inflation stage. We choose the initial spectrum of RGWs at the time  $\eta_i$  of the horizon-crossing [34, 38, 39, 42]

$$h(\nu, \eta_i) = \frac{2k^{3/2}}{\pi} |h_k(\eta_i)| = A \left( \frac{k}{k_H} \right)^{2+\beta_{inf}}, \quad (6)$$

where  $k_H \simeq 2\pi$  is the comoving wavenumber corresponding to the Hubble radius,  $A$  is a  $k$ -independent constant to be normalized by the present observed CMB anisotropies in practice, and the spectral index  $\beta_{inf}$  is a parameter depending on inflationary models. The special case of  $\beta_{inf} = -2$  is the de Sitter expansion of inflation. If the inflationary expansion is driven by a scalar field, then the index  $\beta_{inf}$  is related to the so-called slow-roll parameters,  $\eta$  and  $\epsilon$  [43], as  $\beta_{inf} = -2 + (\eta - 3\epsilon)$ .  $\beta_{inf}$  is related to the spectral index  $n_S$  of primordial scalar perturbations as  $n_S = 2\beta_{inf} + 5$ . In literature, the RGWs spectrum is also written in the following form [1] [2] [44]

$$\Delta_h^2(k) = A_T \left( \frac{k}{k_0} \right)^{n_T} = \frac{1}{8} h^2(\nu, \eta_i), \quad (7)$$

where the tensor spectrum index  $n_T = 2(\beta_{inf} + 2) \sim 0$  without the running index,  $k_0$  is some pivot wavenumber, taken as  $k_0 = 0.002 \text{ Mpc}^{-1}$  in our calculation, and the tensor spectrum amplitude  $A_T = 2.95 \times 10^{-9} A(k_0) r$ , where  $A(k_0)$  is the scalar power spectrum amplitude that can be determined by the WMAP observations [1, 3, 4], and we take  $A(k_0) \sim 0.8$  accordingly. The tensor/scalar ratio  $r$  is model-dependent, and frequency-dependent [33, 45]. Recently, the 5-year WMAP data improves the upper limit to  $r < 0.43$  (95% CL) [8], and combined with BAO and SN gives  $r < 0.2$  (95% CL) [5] [7]. In our treatment, for simplicity,  $r \simeq 0.37$  is only taken as a constant parameter for normalization of RGWs, except otherwise mentioned.

The resulting functions  $h_k(\eta)$  and  $\dot{h}_k(\eta)$  serve as the tensorial source to CMB anisotropies and polarization. Without reionization, only RGWs  $h_k(\eta_d)$  and  $\dot{h}_k(\eta_d)$  at the decoupling time  $\eta_d$  are relevant, contributing to the primary CMB spectra. When reionization comes,  $h_k(\eta_r)$  and  $\dot{h}_k(\eta_r)$  at the reionization  $\eta_r$  contribute too, mainly contributing to the very large angular reionization bumps of CMB spectra. In Fig.1,  $h_k(\eta_d)$  and  $\dot{h}_k(\eta_d)$ , and  $h_k(\eta_r)$  and  $\dot{h}_k(\eta_r)$  are plotted. The right panel of Fig. 1 shows that,  $\dot{h}_k(\eta_d)$  has the greatest amplitude around  $k \sim 25$ ,

Figure 1: The RGWs  $h_k(\eta_d)$  and  $\dot{h}_k(\eta_d)$  at the decoupling and  $h_k(\eta_r)$  and  $\dot{h}_k(\eta_r)$  at the reionization.

forming a deep trough, whereas  $\dot{h}(\eta_r)$  has the greatest amplitude around  $k \sim 2$ , forming a deep trough. The left panel shows that both  $h_k(\eta_d)$  and  $h_k(\eta_r)$  have similar slope for small  $k$ . As we will see, these features of RGWs at  $\eta_d$  and at  $\eta_r$  are responsible for the profiles of CMB spectra  $C_l^{XX}$ .

### 3. Visibility Function

In Basko and Ponarev's method, the Boltzmann equation of the photon gas for the  $k$ -mode is written as a set of two coupled differential equations [20, 21]

$$\dot{\xi}_k + [ik\mu + q]\xi_k = \dot{h}_k, \quad (8)$$

$$\dot{\beta}_k + [ik\mu + q]\beta_k = qG_k. \quad (9)$$

where  $\beta_k$  is the linear polarization contributed only by linearly polarized CMB photons,

$$\alpha_k \equiv \xi_k - \beta_k \quad (10)$$

is the anisotropy of radiation intensity contributed by both unpolarized (natural light) and polarized CMB photons,  $\mu = \cos \theta$ ,  $q$  is the differential optical depth, and

$$G_k(\eta) = \frac{3}{16} \int_{-1}^1 d\mu' [(1 + \mu'^2)^2 \beta_k - \frac{1}{2}(1 - \mu'^2)^2 \xi_k]. \quad (11)$$

In the following, we omit the subscript  $k$  for simplicity of notation. The formal solutions of Eqs.(8) and (9) at any time  $\eta$  can be written as the following time integrations [33, 34]:

$$\xi(\eta) = \int_0^\eta \dot{h}(\eta') e^{-\kappa(\eta, \eta')} e^{ik\mu(\eta' - \eta)} d\eta', \quad (12)$$

$$\beta(\eta) = \int_0^\mu G(\eta') q(\eta') e^{-\kappa(\eta, \eta')} e^{ik\mu(\eta' - \eta)} d\eta', \quad (13)$$

where

$$\kappa(\eta', \eta) \equiv \int_\eta^{\eta'} q d\eta = \kappa(\eta) - \kappa(\eta') \quad (14)$$

with the optical depth given by

$$\kappa(\eta) \equiv \kappa(\eta_0, \eta) = \int_\eta^{\eta_0} q(\eta') d\eta' \quad (15)$$

from the present time  $\eta_0$  back to an earlier time  $\eta$ , such that

$$q(\eta) = -\frac{d\kappa(\eta)}{d\eta}. \quad (16)$$

The CMB anisotropies and polarization are usually expressed in terms of their Legendre components

$$\xi_l(\eta) = \frac{1}{2} \int_{-1}^1 d\mu \xi(\eta, \mu) P_l(\mu), \quad (17)$$

$$\beta_l(\eta) = \frac{1}{2} \int_{-1}^1 d\mu \beta(\eta, \mu) P_l(\mu), \quad (18)$$

where  $P_l$  is the Legendre function. By the expansion formula

$$e^{ix\mu} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(x) P_l(\mu) \quad (19)$$

and the ortho-normal relation for the Legendre functions, the components at the present time  $\eta_0$  are given by the following

$$\xi_l(\eta_0) = i^l \int_0^{\eta_0} e^{-\kappa(\eta)} \dot{h}(\eta) j_l(k(\eta - \eta_0)) d\eta, \quad (20)$$

$$\beta_l(\eta_0) = i^l \int_0^{\eta_0} G(\eta) V(\eta) j_l(k(\eta - \eta_0)) d\eta, \quad (21)$$

where

$$V(\eta) = q(\eta) e^{-\kappa(\eta)} \quad (22)$$

is the visibility function. As one sees, to analytically carry out the integrations in Eqs.(20) and (21), one needs the explicit expression of  $e^{-\kappa(\eta)}$  and  $V(\eta)$ , which are determined by the whole history of ionization. In the following we will give approximate formula of both functions.

$V(\eta)$  has the meaning of the probability that a CMB photon reaching us today was last scattered by free electrons at the time  $\eta$ . Without the reionization,  $V(\eta)$  would have only one sharp peak around  $z \sim 1100$  for the decoupling, and satisfies the normalization condition

$$\int_0^{\eta_0} V(\eta) d\eta = 1. \quad (23)$$

When the reionization is included,  $V(\eta)$  will have, around  $z \sim 11$ , another peak. If the universe was reionized twice, say at  $z \sim 6$  and  $z \sim 16$ , [46, 47, 48],  $V(\eta)$  would have double peaks for reionization. We consider only the case of a single reionization in this paper. Then, as a function of  $\eta$ ,  $V(\eta)$  is mainly distributed around decoupling and reionization, and is effectively



Figure 2: The visibility function  $V_d(\eta)$  for the decoupling around  $z \sim 1100$ . Both the analytic and the fitting by two half-gaussian functions are shown.

vanishing in the region far away from the peaks, as shown in the Panel (d) in Fig.6. Thus the time integration of Eq.(23) can be practically split into two parts

$$\int_0^{\eta_{split}} V_d(\eta) d\eta + \int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta = 1, \quad (24)$$

where  $V_d(\eta)$  and  $V_r(\eta)$  are the portions of  $V(\eta)$  for decoupling and reionization, respectively, and  $\eta_{split}$  is some point between decoupling and reionization with  $V(\eta_{split}) \simeq 0$ . In calculation we can take, say,  $\eta_{split} = 0.297$  corresponding to a redshift  $z \simeq 100$ . In Eq.(24),  $\int_0^{\eta_{split}} V_d(\eta) d\eta$  is the area covered under the curve of  $V_d(\eta)$ , and stands for the probability that a photon was last scattered during the decoupling. Similarly,  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta$  is the probability that a photon was last re-scattered during the reionization, i.e., the amount of CMB photons out of the total that are rescattered. According to Eq.(24), their sum is constrained to be unity. This has a physical interpretation: more CMB photons are last scattered around  $\sim \eta_r$ , less will be last scattered around  $\sim \eta_d$ . During reionization the intrinsic anisotropies of this portion of CMB photons were washed out, and new polarizations were generated on large angular scales. As we will see,  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta$  depends essentially on the optical depth up to the reionization.

Now let us specify the visibility functions  $V_d(\eta)$  and  $V_r(\eta)$ . First the decoupling process is better understood, whose  $V_d(\eta)$  has been given explicitly, which depends the baryon fraction  $\Omega_B$  [19, 49, 50]. As a function of time, the profile of  $V_d(\eta)$  itself looks like a sharp peak around the decoupling  $z \sim 1100$ . Thus, when it appears as a factor of the integrand in the time integration (21) for the polarization  $\beta_l(\eta_0)$ , it actually plays a filtering role: only the narrow time range around the decoupling contributes substantially to the integral of Eq.(21). To facilitate analytic calculations of CMB polarization,  $V_d(\eta)$  has been approximated by the following two pieces of half gaussian function [33, 34]

$$V_d(\eta) = \begin{cases} V(\eta_d) \exp\left(-\frac{(\eta-\eta_d)^2}{2\Delta\eta_{d1}^2}\right), & (\eta \leq \eta_d), \\ V(\eta_d) \exp\left(-\frac{(\eta-\eta_d)^2}{2\Delta\eta_{d2}^2}\right), & (\eta > \eta_d), \end{cases} \quad (25)$$

where  $\eta_d$  is the decoupling time, which is taken  $\eta_d = 0.0707$  corresponding to a redshift  $z_d = 1100$ ,  $\Delta\eta_{d1} = 0.00639$ ,  $\Delta\eta_{d2} = 0.0117$ , and  $(\Delta\eta_{d1} + \Delta\eta_{d2})/2 = \Delta\eta_d = 0.00905$  is the thickness of the decoupling. Eq.(25) improves a single gaussian function [32] by  $\sim 10\%$  in accuracy and at the same time allows an analytic treatment of the CMB polarization spectrum. We have

Figure 3: The three models of reionization with a fixed optical depth  $\kappa_r = 0.084$ . For each  $X_e(\eta)$  given in Eqs.(26), (27), and (28), the functions  $q_r(\eta)$ ,  $\kappa_r(\eta)$ , and  $V_r(\eta)$  are calculated according to the formulae in Eqs.(29), (30), and (31), respectively.

checked that the errors between Eq.(25) and the numerical formulae given in [19, 50] is very small,  $\leq 3.9\%$  in the whole range. The coefficient  $V(\eta_d)$ , as the height of  $V_d(\eta)$ , also depends on the reionization through the normalization in Eq.(24). The analytic  $V_d(\eta)$  with  $\Omega_B = 0.046$  and its fitting are shown in Fig.2.

Next, understanding of the reionization as a physical process is still underway, and various tentative models have been proposed for it. Spatially, the reionization might have occurred inhomogeneously [51, 52, 53, 54, 55], resulting in modifications on the small angular scales part of CMB spectra. Models of double reionization [46, 47], or its variants, such as peak-like reionization [56], have also been proposed. In the following, we will work with three simple homogeneous models, whose ionization fraction  $X_e(\eta)$  are explicitly given. One is the sudden reionization model with

$$X_e(\eta) = \begin{cases} 0, & \text{for } \eta < \eta_r, \\ 1, & \text{for } \eta \geq \eta_r, \end{cases} \quad (26)$$

where  $\eta_r$  is the reionization time. For concreteness of illustration, in our calculation we take  $\eta_r = 0.915$ , corresponding to the redshift  $z_r = 11$ . This is the simplest model often used in the literature. But there are accumulating evidence that the reionization is an extended process, stretching from  $z \simeq 6$  up to  $z \sim 11$ , even up to as early as  $z \sim 20$  [8, 57, 58]. For instance, studies of Ly $\alpha$  Gunn-Peterson absorption [59] indicate a rapid increase in the ionized fraction of the intergalactic medium at a redshift lower than  $z_r \simeq 6$ . On the other hand, the WMAP observations of CMB found a much earlier reionization,  $z_r = 17 \pm 5$  by WMAP 1-yr [1],  $z_r = 10.9^{+2.7}_{-2.3}$  by WMAP 3-yr [3],  $z_r = 11.0 \pm 1.4$  (68% CL) by WMAP 5-yr [8], and  $z_r = 10.8 \pm 1.4$  by WMAP 5-yr combined with SN and BAO [7] [5]. One extended reionization model is the  $\eta$ -linear reionization with

$$X_e(\eta) = \begin{cases} 0, & \text{for } \eta < \eta_{r1} \\ \frac{\eta - \eta_{r1}}{\eta_{r2} - \eta_{r1}}, & \text{for } \eta_{r1} < \eta < \eta_{r2}, \\ 1, & \text{for } \eta > \eta_{r2}. \end{cases} \quad (27)$$

where  $\eta_{r1}$  and  $\eta_{r2}$  are the beginning and end of reionization. For instance, one can take  $\eta_{r1} = 0.685$  and  $\eta_{r2} = 1.20712$ , corresponding to  $z_{r1} = 20$  and  $z_{r2} = 6$ , respectively. This model is closer to the result of WMAP 5-yr fitted by the two step reionization [8]. Another extended

reionization model is the  $z$ -linear model with [55]:

$$X_e(z) = \begin{cases} 0, & \text{for } z > z_{r1} \\ 1 - \frac{z - z_{r2}}{z_{r1} - z_{r2}}, & \text{for } z_{r1} > z > z_{r2}, \\ 1, & \text{for } z \leq z_{r2}. \end{cases} \quad (28)$$

For  $z_{r1} = 20$  and  $z_{r2} = 6$ , one has  $X_e(z) = 1 - (z - 6)/14$ . The ionization fraction  $X_e(\eta)$  for these three reionization models are comparatively shown in Fig.3.

Given  $X_e(\eta)$  in the above three models, the differential optical depth for reionization can be directly calculated by the formula [19, 55, 60]:

$$q_r(\eta) = C_c \frac{a(\eta_0)^3}{a(\eta)^2} X_e(\eta), \quad (29)$$

where the constant  $C_c = (1 - \frac{Y_p}{2}) \frac{\Omega_b \rho_c \sigma_T}{m_p}$ ,  $Y_p \simeq 0.23$  is the primordial helium fraction,  $\sigma_T$  is the cross section of Thompson scattering,  $m_p$  is the mass of a proton. For  $\Omega_b = 0.045$ ,  $C_c \simeq 0.142 \times 10^{-28} \text{ m}^{-1}$ . Since the value of  $Y_p$  from observations has considerable large error bars [61], in our treatment  $C_c$  is allowed to vary slightly around this value. From Eq.(15) follows the optical depth for reionization as an integration

$$\kappa_r(\eta) = \int_{\eta}^{\eta_0} q_r(\eta') d\eta', \quad (30)$$

and, from Eq.(22) follows the visibility function for the reionization,

$$V_r(\eta) = q_r(\eta) e^{-\kappa_r(\eta)}. \quad (31)$$

For instance, for the sudden reionization model, one easily obtains

$$\begin{aligned} \kappa_r(\eta) = & \frac{C_c}{3} \frac{l_H^3}{a_m^2} \left[ (\eta - \eta_m)^{-3} - (\eta_E - \eta_m)^{-3} \right] \\ & + \frac{C_c}{2\gamma + 1} l_H \left[ (\eta_a - \eta_0)^{2\gamma+1} - (\eta_a - \eta_E)^{2\gamma+1} \right], \quad (\eta \geq \eta_r), \end{aligned} \quad (32)$$

where all the parameters have been given bellow Eq.(5). For a reionization model, the most important quantity  $\kappa_r \equiv \kappa_r(\eta_b)$  is the value of the optical depth from  $\eta_0$  back up to some time  $\eta_b$  before the reionization, where  $q_r(\eta_b)$  is practically vanishing. For example, one can take  $\eta_b = \eta_{split}$ . In practice, one can conveniently take  $\eta_b = \eta_r$  for the sudden model, and take  $\eta_b = 0.5$  for the  $\eta$ -linear and  $z$ -linear models.  $\kappa_r$  is an integral constraint on the reionization history. On the observational side, based upon treatments of a sudden model, WMAP 1-yr gives  $\kappa_r = 0.17 \pm 0.04$  [1], and WMAP 3-yr gives  $\kappa_r = 0.09 \pm 0.03$  [3], and WMAP 5-yr gives

Figure 4: The  $\eta$ -linear reionization model with  $\kappa_r = 0.084$ . The solid lines are the calculated results. The dashed lines are the fitting by two half Gaussian functions in Eq.(33).

Figure 5: The  $z$ -linear reionization model and its fitting.

$\kappa_r = 0.087 \pm 0.017$  [8], and WMAP 5-yr combined with SN and BAO yields  $\kappa_r = 0.084 \pm 0.016$  [5, 7]. To be specific in calculation, we will take the value  $\kappa_r = 0.084$  for all three reionization models in this paper, except when it is mentioned otherwise. However, note that, for extended reionization models, one should be careful in applying the WMAP observed value of  $\kappa_r$ , as it is obtained by using a sudden model. For the  $\eta$ -linear model with  $X_e(\eta)$  given in Eq.(27), one uses the formulae of Eqs.(29) (30) (31) to compute  $q_r(\eta)$ ,  $\kappa_r(\eta)$ ,  $V_r(\eta)$ . For the  $z$ -linear model with  $X_e(\eta)$  in Eq.(28), one does similar computations. The resulting  $q_r(\eta)$ ,  $\kappa_r(\eta)$ , and  $V_r(\eta)$  for these three models are plotted in Fig.3.

The value of optical depth  $\kappa_r$  determines the area  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta$  introduced in Eq.(24). For a fixed  $\kappa_r = 0.084$ , the integration of Eq.(31) yields  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta = 0.0795$  in the sudden model,  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta = 0.07953$  in the  $\eta$ -linear model, and  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta = 0.07973$  in the  $z$ -linear model, respectively. So two gradual models have slightly larger area than the sudden model. Besides, our computations also show that a larger  $\kappa_r$  yields a larger  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta$  and a smaller  $\int_0^{\eta_{split}} V_d(\eta) d\eta$  due to Eq.(24), meaning that a CMB photon reaching us was more likely last scattered at reionization. As we shall see explicitly, for CMB spectra, this will enhance the reionization bumps on large scales and reduce the primary peaks due decoupling.

To facilitate analytical calculations of CMB polarization, similar to the treatments of  $V_d(\eta)$  for the decoupling,  $V_r(\eta)$  can be also approximated by some fitting formula. For the  $\eta$ -linear model, it is fitted by the following two pieces of half Gaussian functions

$$V_r(\eta) = \begin{cases} V(\eta_r) \exp\left(-\frac{(\eta-\eta_r)^2}{2(\Delta\eta_{r1})^2}\right), & (\eta < \eta_r), \\ V(\eta_r) \exp\left(-\frac{(\eta-\eta_r)^2}{2(\Delta\eta_{r2})^2}\right), & (\eta > \eta_r), \end{cases} \quad (33)$$

where  $\Delta\eta_{r1} = 0.147$ ,  $\Delta\eta_{r2} = 0.425$ ,  $\Delta\eta_r = (\Delta\eta_{r1} + \Delta\eta_{r2})/2 = 0.286$ , and  $\eta_r = 0.935$  ( $z_r = 10.5$ ). It is plotted in Panel (c) of Fig.4 under the requirement that it gives the same area  $\int_{\eta_{split}}^{\eta_0} V_r(\eta) d\eta$  as the calculated one. For the  $z$ -linear model, the fitting formula is similar to Eq.(33) but with the parameters  $\Delta\eta_{r1} = 0.100$ ,  $\Delta\eta_{r2} = 0.366$ ,  $\Delta\eta_r = (\Delta\eta_{r1} + \Delta\eta_{r2})/2 = 0.233$ , and  $\eta_r = 0.855$  ( $z_r = 13$ ). It is plotted in Panel (c) of Fig. 5. Here for the two extended models, the value of  $\eta_r$  has been taken to correspond to the maximum of  $V_r(\eta)$ . For the sudden model,

Figure 6: The sudden reionization model and its fitting. Panel (c) shows that the fitting  $V_r(\eta)$  by Eq.(34) has large errors to the calculated one. The evolution history of  $V(\eta)$ , including both reionization and decoupling, is sketched in Panel (d).

it can be fitted by a half piece of Gaussian function

$$V_r(\eta) = \begin{cases} 0, & (\text{for } \eta < \eta_r), \\ V(\eta_r) \exp\left(-\frac{(\eta-\eta_r)^2}{2(\Delta\eta_{dr})^2}\right), & (\text{for } \eta > \eta_r), \end{cases} \quad (34)$$

with the width  $\Delta\eta_{dr} = 0.247$ , plotted in Panel (c) of Fig.6. The half-gaussian fitting of  $V_r(\eta)$  for the sudden model is not as accurate as those for the two extended models. It should be expected that in the sudden model the analytical CMB spectra  $C_l^{XX}$  based on its fitting formula (34) is not as good as those in the two extended models.

We mention that, given a fixed  $\kappa_r$ , the respective height  $V(\eta_r)$  in Eqs.(26), (34), and (33) are also determined automatically. From these fitting  $V_r(\eta)$ , one can convert it to obtain the corresponding optical functions

$$e^{-\kappa_r(\eta)} = 1 - \int_{\eta}^{\eta_0} V_r(\eta) d\eta, \quad (35)$$

$$\kappa_r(\eta) = -\ln\left(1 - \int_{\eta}^{\eta_0} V_r(\eta) d\eta\right), \quad (36)$$

$$q_r(\eta) = \frac{V_r(\eta)}{\left(1 - \int_{\eta}^{\eta_0} V_r(\eta) d\eta\right)}. \quad (37)$$

It should be mentioned that the approximate fitting of  $V_r(\eta)$  by Eq.(33) underestimates the value of  $V_r$  in the range  $\eta > \eta_r$  by  $\sim 9.1\%$ . For the  $z$ -linear model, the fitting by half Gaussian functions underestimates the value of  $V_r$  in the range  $\eta > \eta_r$  by  $\sim 8.6\%$ . However, this kind of error of the fitting can partially compensated in treating the damping factors occurring in the time integration of the polarization mode, as will be given in the following. The gaussian fitting of Eq.(34) for the sudden model is included only for illustration purpose, as its error is larger than the two extended models.

#### 4. Spectra of CMB Anisotropies and Polarization

By applying the same kind of approximate integration technique as in Refs.[33, 34], up to the second order of a small  $1/q^2$  in the tight coupling limit, the function  $G(\eta)$  in Eq.(11) can be written as

$$G(\eta) = -\frac{1}{10} \int_0^{\eta} \dot{h}(\eta') e^{-\frac{3}{10}\kappa(\eta') - \frac{7}{10}\kappa(\eta)} d\eta', \quad (38)$$

and the integration of polarization mode in Eq.(21) is written as

$$\beta_l(\eta_0) = -\frac{1}{10}i^l \int_0^{\eta_0} d\eta V(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \int_0^\eta d\eta' e^{-\frac{3}{10}\kappa(\eta') - \frac{7}{10}\kappa(\eta)}. \quad (39)$$

Since the visibility function  $V(\eta)$  for the whole history consists of two effectively non-overlapping functions,  $V_d(\eta)$  and  $V_r(\eta)$ , the  $\eta$ -time integration  $\int_0^{\eta_0} d\eta$  in the above is naturally split into a sum of two integrations:

$$\begin{aligned} \beta_l(\eta_0) = & -\frac{1}{10}i^l \int_0^{\eta_{split}} d\eta V_d(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \int_0^\eta d\eta' e^{-\frac{3}{10}\kappa(\eta') - \frac{7}{10}\kappa(\eta)} \\ & -\frac{1}{10}i^l \int_{\eta_{split}}^{\eta_0} d\eta V_r(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \int_0^\eta d\eta' e^{-\frac{3}{10}\kappa(\eta') - \frac{7}{10}\kappa(\eta)}. \end{aligned} \quad (40)$$

One defines the integration variable  $x \equiv \kappa(\eta')/\kappa(\eta)$  to replace the variable  $\eta'$  in the above. Since  $V_d(\eta)$  is peaked around  $\eta_d$  with a width  $\Delta\eta_d$ , and, similarly,  $V_r(\eta)$  is peaked around  $\eta_r$  with a width  $\Delta\eta_r$ , one can take  $d\eta' \simeq -\Delta\eta_d \frac{dx}{x}$  and  $d\eta' \simeq -\Delta\eta_r \frac{dx}{x}$  as approximation, respectively.

$$\begin{aligned} \beta_l(\eta_0) = & -\frac{1}{10}i^l \Delta\eta_d \int_0^{\eta_{split}} d\eta V_d(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \int_1^\infty \frac{dx}{x} e^{-\frac{3}{10}\kappa(\eta)x - \frac{7}{10}\kappa(\eta)} \\ & -\frac{1}{10}i^l \Delta\eta_r \int_{\eta_{split}}^{\eta_0} d\eta V_r(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \int_1^\infty \frac{dx}{x} e^{-\frac{3}{10}\kappa(\eta)x - \frac{7}{10}\kappa(\eta)}. \end{aligned} \quad (41)$$

For each term in the above, the  $\eta$ -time integration can be dealt with, using the same kind of treatment as in Ref.[33, 34]. For the decoupling one has

$$\int_0^{\eta_{split}} d\eta V_d(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \simeq D_d(k) \dot{h}(\eta_d) j_l(k(\eta_d - \eta_0)) \int_0^{\eta_{split}} d\eta V_d(\eta), \quad (42)$$

where the damping factor for the decoupling is given by the following fitting formula

$$D_d(k) = \frac{1.4}{2} [e^{-c(k\Delta\eta_{d1})^b} + e^{-c(k\Delta\eta_{d2})^b}], \quad (43)$$

which can be simplified by

$$D_d(k) = 1.4e^{-c(k\Delta\eta_d)^b}, \quad (44)$$

with  $c$  and  $b$  being two fitting parameters. For CMB spectra without reionization, it has been shown in Ref.[34] that both damping factors in Eqs.(43) and (44)  $c \simeq 0.6$  and  $b \simeq 0.85$  give a good match with the numerical result by CAMB [16] over an extended range  $l \leq 600$ , covering the first three primary peaks, and the error is only  $\sim 3\%$ .

Similarly, the  $\eta$ -time integration for the reionization is

$$\int_{\eta_{split}}^{\eta_0} d\eta V_r(\eta) \dot{h}(\eta) j_l(k(\eta - \eta_0)) \simeq D_r(k) \dot{h}(\eta_r) j_l(k(\eta_r - \eta_0)) \int_{\eta_{split}}^{\eta_0} d\eta V_d(\eta), \quad (45)$$

where the damping factor for the extended models is taken to be

$$D_r(k) = \frac{1.4}{2} [e^{-c(k\Delta\eta_{r1})^b} + e^{-c(k\Delta\eta_{r2})^b}], \quad (46)$$

or for the sudden reionization

$$D_r(k) = \frac{1.4}{2} e^{-c(k\Delta\eta_r)^b}. \quad (47)$$

Here the parameter  $c$  and  $b$  in Eqs.(46) and (47) for reionization could take values different from those for decoupling. For simplicity, we let them take the values that are the same as in  $D_d(k)$ . Guided by the error estimation for the decoupling case, we can only estimate the errors due to  $D_r(k)$  in Eq.(46) for the two extended models upon the reionization bumps of polarization spectra to be  $\leq 10\%$ , the same order of magnitude as those of the fitting  $V_r(\eta)$  in Eq.(33).

Substituting Eqs.(42) and (45) into Eq.(41), and performing the integrations  $\int d\eta$  first,

$$\int_0^{\eta_{split}} d\eta V_d(\eta) e^{-\frac{3}{10}\kappa(\eta)x - \frac{7}{10}\kappa(\eta)} = \int_{\kappa_r}^{\infty} d\kappa e^{-\frac{3}{10}\kappa x - \frac{17}{10}\kappa} = \frac{1}{\frac{17}{10} + \frac{3}{10}x} e^{-(\frac{17}{10} + \frac{3}{10}x)\kappa_r} \quad (48)$$

$$\int_{\eta_{split}}^{\eta_0} d\eta V_r(\eta) e^{-\frac{3}{10}\kappa(\eta)x - \frac{7}{10}\kappa(\eta)} = \int_0^{\kappa_r} d\kappa e^{-\frac{3}{10}\kappa x - \frac{17}{10}\kappa} = \frac{1}{\frac{17}{10} + \frac{3}{10}x} [1 - e^{-(\frac{17}{10} + \frac{3}{10}x)\kappa_r}] \quad (49)$$

one finally obtains the expression of the polarization mode as a sum of two parts

$$\beta_l(\eta_0) = -\frac{1}{10} i^l \left[ A_1(\kappa_r) D_d(k) \Delta\eta_d \dot{h}(\eta_d) j_l(k(\eta_d - \eta_0)) + A_2(\kappa_r) D_r(k) \Delta\eta_r \dot{h}(\eta_r) j_l(k(\eta_r - \eta_0)) \right] \quad (50)$$

where the  $\kappa_r$ -dependence coefficients

$$A_1(\kappa_r) = \int_1^{\infty} \frac{dx}{x(\frac{17}{10} + \frac{3}{10}x)} e^{-(\frac{17}{10} + \frac{3}{10}x)\kappa_r}, \quad (51)$$

$$A_2(\kappa_r) = \int_1^{\infty} \frac{dx}{x(\frac{17}{10} + \frac{3}{10}x)} [1 - e^{-(\frac{17}{10} + \frac{3}{10}x)\kappa_r}], \quad (52)$$

both being independent of the wavenumber  $k$ , and the sum is  $A_1(\kappa_r) + A_2(\kappa_r) = \frac{10}{17} \ln \frac{20}{3} \simeq 1.116$ , independent of  $\kappa_r$ . If one sets  $A_2 = 0$  and  $A_1 = \frac{10}{17} \ln \frac{20}{3}$ , Eq.(50) reduces to exactly that of the non-reionization case [33, 34]. Actually, after the sum is normalized to unity, the two coefficients have the physical meaning:

$$a_1(\kappa_r) \equiv \frac{A_1(\kappa_r)}{\frac{10}{17} \ln \frac{20}{3}} \quad (53)$$

is the probability that a polarized photon we perceive was last scattered during the decoupling epoch, and

$$a_2(\kappa_r) \equiv \frac{A_2(\kappa_r)}{\frac{10}{17} \ln \frac{20}{3}} \quad (54)$$

Figure 7: The normalized coefficients  $a_1(\kappa_r)$  and  $a_2(\kappa_r)$  of the polarization  $\beta_l$ . A larger  $\kappa_r$  yields lower  $a_1(\kappa_r)$  and higher  $a_2(\kappa_r)$ , i.e., will yield lower primary peaks and higher reionization bump in  $C_l^{EE}$  and  $C_l^{BB}$ . Also plotted are the coefficients  $e^{-\kappa_r}$  and  $(1 - e^{-\kappa_r})$  of the temperature anisotropies  $\alpha_l$ . Notice that  $a_1(\kappa_r)$  and  $a_2(\kappa_r)$  vary with  $\kappa_r$  more drastically than  $e^{-\kappa_r}$  and  $(1 - e^{-\kappa_r})$ , respectively.

is the probability that a polarized photon we perceive was last scattered during the time interval from the beginning of reionization up to the present time  $\eta_0$ . It is found that  $a_1(\kappa_r)$  is a decreasing function of  $\kappa_r$  and  $a_2(\kappa_r)$  is an increasing one, as shown in Fig. 7. Therefore, if more CMB photons are scattered by the free electrons during the reionization, the optical depth  $\kappa_r$  acquires a larger value, giving rise to a higher coefficient  $A_2(\kappa_r)$  and, at the same time, a lower coefficient  $A_1(\kappa_r)$ . The  $A_1(\kappa_r)$  part in  $\beta_l$  from the decoupling will give rise to the primary peaks of  $C_l^{EE}$  and  $C_l^{BB}$ , and will be prominent on small angular scales with  $l \geq 100$ . The  $A_2(\kappa_r)$  part from the reionization will be dominant on large angular scales and will yield the reionization bumps of  $C_l^{EE}$  and  $C_l^{BB}$  around  $l < 10$ .

The analytical expression (50) has the merit that effects of relevant physical elements upon the polarization have been explicitly isolated and displayed. The  $\kappa_r$ -dependence of  $\beta_l$  is attributed to the coefficients  $A_1(\kappa_r)$  and  $A_2(\kappa_r)$ , which determine the relative heights of the primary peaks and the reionization bump. Other effects of reionization is encoded in the factor  $D_r(k)\Delta\eta_r$ . The effects of decoupling are absorbed in  $D_d(k)\Delta\eta_d$ . The effect of RGWs upon the polarization are given by the time derivatives  $\dot{h}(\eta_d)$  at  $\eta_d$  and  $\dot{h}(\eta_r)$  at  $\eta_r$ , which not only contain the cosmological information, such as inflation and NFS, etc., more importantly, but also determine the overall profiles of  $C_l^{EE}$  and  $C_l^{BB}$ , such as the locations of peaks and troughs, and of bumps. The factors  $j_l(k(\eta_d - \eta_0))$  and  $j_l(k(\eta_r - \eta_0))$  just play the role of conversion from the wavenumber  $k$ -space into the multipole  $l$ -space.

To calculate the temperature anisotropies, we need to evaluate  $\xi_l$  in Eq.(20), which contains the factor  $e^{-\kappa(\eta)}$ . This also needs to be dealt with properly. As shown in Fig.6, the factor  $e^{-\kappa(\eta)}$  has two steps, one at the decoupling  $\eta = \eta_d$ , and another at  $\eta \simeq \eta_r$  caused by the reionization. It can be approximated by the following two-step function

$$e^{-\kappa(\eta)} \simeq \begin{cases} 0 & (\eta < \eta_d); \\ e^{-\kappa_r} & (\eta_d < \eta < \eta_r); \\ 1 & (\eta_r < \eta < \eta_0), \end{cases} \quad (55)$$

and its reionization-relevant part  $e^{-\kappa_r(\eta)}$  is between  $(\eta_d, \eta_0)$ . By Eq.(35),  $e^{-\kappa_r(\eta)}$  is the integration of  $V_r(\eta)$  from  $\eta$  to  $\eta_0$ , determined by the area under the curve of  $V_r(\eta)$ , not very sensitive to the



detailed shape of  $V_r(\eta)$ . Therefore, the approximate formula (55) will be used for the three models of reionization, with their respective values of  $\eta_r$ . Note that Eq.(55) tends to overestimate the contribution of the reionization to the integration, since  $e^{-\kappa(\eta)}$  shown in Fig.6 increases gradually from  $e^{-\kappa_r}$  at  $\eta_r$  up to 1 for  $\eta \gg \eta_r$ , instead of instantaneously jumping up to 1 at  $\eta_r$ . To compensate this overestimation, in actually calculating  $\xi_l(\eta_0)$  in the linear model, we may use the value of  $\eta_r$  slightly greater than 0.935. But this adjustment of the time  $\eta_r$  does not apply to  $\beta_l(\eta_0)$  in Eq.(50). Substituting Eq.(55) into Eq.(20), the integration for  $\xi_l$  is split into two terms

$$\xi_l(\eta_0) \simeq i^l \int_{\eta_d}^{\eta_r} e^{-\kappa_r} \dot{h}(\eta) j_l(k(\eta_0 - \eta)) d\eta + i^l \int_{\eta_r}^{\eta_0} \dot{h}(\eta) j_l(k(\eta_0 - \eta)) d\eta. \quad (56)$$

Following the similar treatments in [45, 34], each term is integrated by parts, yielding the following approximate expression

$$\xi_l(\eta_0) = -i^l \left[ e^{-\kappa_r} h(\eta_d) j_l(k(\eta_0 - \eta_d)) + (1 - e^{-\kappa_r}) h(\eta_r) j_l(k(\eta_0 - \eta_r)) \right], \quad (57)$$

where the first term is generated by  $h(\eta_d)$  at the recombination and the second term is due to  $h(\eta_r)$  at the reionization. Eq.(10) then yields the mode of CMB temperature anisotropies  $\alpha_l(\eta_0) = \xi_l(\eta_0) - \beta_l(\eta_0)$ . In fact,  $\alpha_l(\eta_0)$  is essentially contributed by  $\xi_l(\eta_0)$  since the amplitude of  $\xi_l(\eta_0)$  is about two orders higher than that of  $\beta_l(\eta_0)$ . Writing down explicitly, one has the approximate, analytic expression of the mode of CMB temperature anisotropies, including the reionization,

$$\begin{aligned} \alpha_l(\eta_0) = & -i^l j_l(k(\eta_0 - \eta_d)) \left[ e^{-\kappa_r} h(\eta_d) - \frac{1}{10} A_1(\kappa_r) D_d(k) \Delta \eta_d \dot{h}(\eta_d) \right] \\ & - i^l j_l(k(\eta_0 - \eta_r)) \left[ (1 - e^{-\kappa_r}) h(\eta_r) - \frac{1}{10} A_2(\kappa_r) D_r(k) \Delta \eta_r \dot{h}(\eta_r) \right]. \end{aligned} \quad (58)$$

In this expression, the first term containing  $h(\eta_d)$  and  $\dot{h}(\eta_d)$  is brought by the decoupling and responsible for the primary peaks, whereas the last term containing  $h(\eta_r)$  and  $\dot{h}(\eta_r)$  is brought in by reionization and prominent on large angle scales with  $l < 10$ . When one sets  $A_1 = 1$ ,  $A_2 = 0$ , and  $e^{-\kappa_r} = 1$ , Eq.(58) reduces to the results for the case without reionization [34]. The  $\kappa_r$ -dependence of  $\alpha_l$  is mainly attributed to the factors  $e^{-\kappa_r}$  and  $(1 - e^{-\kappa_r})$ , while the portion containing  $A_1(\kappa_r)$  and  $A_2(\kappa_r)$  is the subdominant  $\beta_l$ . By Eq.(35) and the definition of  $\kappa_r$ , on has

$$e^{-\kappa_r} = 1 - \int_{\eta_b}^{\eta_0} V_r(\eta) d\eta, \quad (59)$$

which has a physical interpretation: the probability of a CMB photon being last scattered during the earlier epoch before the reionization. Since  $e^{-\kappa_r} < 1$  for  $\kappa_r > 0$ , it will cause a slight decrease

in the amplitude of the temperature anisotropies, as demonstrated in Eq.(58). Correspondingly, the factor  $(1 - e^{-\kappa_r})$  in front of  $h(\eta_r)$  is

$$1 - e^{-\kappa_r} = \int_{\eta_b}^{\eta_0} V_r(\eta) d\eta, \quad (60)$$

recognized as the probability of a CMB photon being last scattered during the time interval from the reionization up to the present time  $\eta_0$ . These foregoing probabilistic interpretations have the parallels in the case of CMB anisotropies generated by scalar perturbations, where reionization also brings about a similar exponential factors  $e^{-\kappa_r}$  in the temperature anisotropies, and a physical illustration on its appearance is given in Ref.[62]. It should be mentioned that the probabilities in Eqs.(59) and (60) are respectively different from the normalized  $a_1(\kappa_r)$  and  $a_2(\kappa_r)$ , the latter are for the polarized photons. Moreover, as shown in Fig. 7,  $a_1(\kappa_r)$  decreases with  $\kappa_r$  much faster than  $e^{-\kappa_r}$  does, and  $a_2(\kappa_r)$  increases much faster than  $(1 - e^{-\kappa_r})$ . In this sense, the polarization  $\beta_l(\eta_0)$  is more sensitive to  $\kappa_r$  than the temperature anisotropies  $\alpha_l(\eta_0)$ . Therefore, one may say that the polarization spectra  $C_l^{EE}$  are  $C_l^{BB}$  are more sensitive probes into the reionization than the temperature anisotropies spectrum  $C_l^{TT}$ .

With  $\alpha_l$  and  $\beta_l$  being ready, one can compute straightforwardly the CMB spectra caused by RGWs. The detailed derivations have been demonstrated in Refs. [27, 33, 34]. In particular, some minor misprints of the coefficients in Ref.[27] have been pointed out and corrected in Refs. [33, 34]. The temperature anisotropies

$$C_l^{TT} = \frac{1}{8\pi} \frac{(l+2)!}{(l-2)!} \int k^2 dk \left| \frac{\alpha_{l-2}(\eta_0)}{(2l-1)(2l+1)} - \frac{2\alpha_l(\eta_0)}{(2l-1)(2l+3)} + \frac{\alpha_{l+2}(\eta_0)}{(2l+1)(2l+3)} \right|^2, \quad (61)$$

the electric type of polarization

$$C_l^{EE} = \frac{1}{16\pi} \int k^2 dk \left| \frac{(l+1)(l+2)\beta_{l-2}(\eta_0)}{(2l-1)(2l+1)} + \frac{6(l-1)(l+2)\beta_l(\eta_0)}{(2l-1)(2l+3)} + \frac{l(l-1)\beta_{l+2}(\eta_0)}{(2l+1)(2l+3)} \right|^2, \quad (62)$$

the magnetic type of polarization

$$C_l^{BB} = \frac{1}{16\pi} \int k^2 dk \left| \frac{2(l+2)\beta_{l-1}(\eta_0)}{(2l+1)} + \frac{2(l-1)\beta_{l+1}(\eta_0)}{(2l+1)} \right|^2, \quad (63)$$

and the temperature-polarization cross

$$C_l^{TE} = \sqrt{\frac{1}{8\pi} \frac{(l+2)!}{(l-2)!}} \sqrt{\frac{1}{16\pi}} \int k^2 dk \left[ \frac{\alpha_{l-2}(\eta_0)}{(2l-1)(2l+1)} - \frac{2\alpha_l(\eta_0)}{(2l-1)(2l+3)} + \frac{\alpha_{l+2}(\eta_0)}{(2l+1)(2l+3)} \right] \times \left[ \frac{(l+1)(l+2)\beta_{l-2}(\eta_0)}{(2l-1)(2l+1)} + \frac{6(l-1)(l+2)\beta_l(\eta_0)}{(2l-1)(2l+3)} + \frac{l(l-1)\beta_{l+2}(\eta_0)}{(2l+1)(2l+3)} \right]. \quad (64)$$

Substituting  $\alpha_l(\eta_0)$  and  $\beta_l(\eta_0)$  into Eqs. (61), (62), (63) and (64) yields the analytical expressions of the spectra of CMB with the modifications of reionization:

$$C_l^{TT} = \frac{1}{8\pi} \frac{(l+2)!}{(l-2)!} \int k^2 dk \left\{ P_{Tl}(k(\eta_0 - \eta_d)) \left[ e^{-\kappa_r} h(\eta_d) - \frac{1}{10} A_1 D_d(k) \Delta \eta_d \dot{h}(\eta_d) \right] \right. \\ \left. + P_{Tl}(k(\eta_0 - \eta_r)) \left[ (1 - e^{-\kappa_r}) h(\eta_r) - \frac{1}{10} A_2 D_r(k) \Delta \eta_r \dot{h}(\eta_r) \right] \right\}^2, \quad (65)$$

$$C_l^{EE} = \frac{1}{16\pi} \left( \frac{1}{10} \right)^2 \int k^2 dk \left[ P_{El}(k(\eta_0 - \eta_d)) A_1 D_d(k) \Delta \eta_d \dot{h}(\eta_d) \right. \\ \left. + P_{El}(k(\eta_0 - \eta_r)) A_2 D_r(k) \Delta \eta_r \dot{h}(\eta_r) \right]^2, \quad (66)$$

$$C_l^{BB} = \frac{1}{16\pi} \left( \frac{1}{10} \right)^2 \int k^2 dk \left[ P_{Bl}(k(\eta_0 - \eta_d)) A_1 D_d(k) \Delta \eta_d \dot{h}(\eta_d) \right. \\ \left. + P_{Bl}(k(\eta_0 - \eta_r)) A_2 D_r(k) \Delta \eta_r \dot{h}(\eta_r) \right]^2, \quad (67)$$

$$C_l^{TE} = -\frac{1}{8\sqrt{2}\pi \times 10} \sqrt{\frac{(l+2)!}{(l-2)!}} \int k^2 dk \\ \frac{1}{2} \left\{ \left[ P_{Tl}(k(\eta_0 - \eta_d)) \left[ e^{-\kappa_r} h(\eta_d) - \frac{1}{10} A_1 D_d(k) \Delta \eta_d \dot{h}(\eta_d) \right] \right. \right. \\ \left. + P_{Tl}(k(\eta_0 - \eta_r)) \left[ (1 - e^{-\kappa_r}) h(\eta_r) - \frac{1}{10} A_2 D_r(k) \Delta \eta_r \dot{h}(\eta_r) \right] \right] \\ \left[ P_{El}(k(\eta_0 - \eta_d)) A_1 D_d(k) \Delta \eta_d \dot{h}(\eta_d) + P_{El}(k(\eta_0 - \eta_r)) A_2 D_r(k) \Delta \eta_r \dot{h}(\eta_r) \right]^* \\ \left. + \text{Complex Conjugate.} \right\} \quad (68)$$

In the above integrations, the projection factors are defined as:

$$P_{Tl}(x) = \frac{j_{l-2}(x)}{(2l-1)(2l+1)} + \frac{2j_l(x)}{(2l-1)(2l+3)} + \frac{j_{l+2}(x)}{(2l+1)(2l+3)} = \frac{j_l(x)}{x^2}, \quad (69)$$

$$P_{El}(x) = \frac{(l+1)(l+2)}{(2l-1)(2l+1)} j_{l-2}(x) - \frac{6(l-1)(l+2)}{(2l-1)(2l+3)} j_l(x) + \frac{l(l-1)}{(2l+1)(2l+3)} j_{l+2}(x) \\ = -\left[ 2 - \frac{l(l-1)}{x^2} \right] j_l(x) + \frac{2}{x} j_{l-1}(x) \quad (70)$$

$$P_{Bl}(x) = \frac{2(l+2)}{(2l+1)} j_{l-1}(x) - \frac{2(l-1)}{(2l+1)} j_{l+1}(x) = 2j_{l-1}(x) - 2\frac{l-1}{x} j_l(x). \quad (71)$$

We apply these formulae to the three reionization models, respectively, and plot the spectra  $C_l^{XX}$ . The reionized spectra  $C_l^{XX}$  are plotted in Fig. 8 for the three models of reionization, in

Figure 8: The spectra  $C_l^{XX}$  in three reionization models. The parameters  $c = 0.65$  and  $b = 0.85$  in  $D_r(k)$  are taken. The optical depth  $\kappa_r = 0.084$  and the ratio  $r = 0.37$  are taken. The numerical result is obtained with the same set of parameters, using CAMB [16].

which we also plot the numerical spectra from the CAMB Online Tool for a comparison [16]. Both the analytic and numerical computation use the same set of parameters  $\kappa_r = 0.084$  and  $r = 0.37$ . On large scales  $l \leq 600$  our analytical  $C_l^{EE}$  and  $C_l^{BB}$  agree with the numerical ones. For the two extended models, the error is  $\sim 3\%$  for the primary peaks, and the error is estimated to be  $\leq 15\%$  for the reionization bumps as superposed from that of decoupling  $\sim 3\%$  and that of reionization  $\sim 10\%$ . Notice also that the analytical  $C_l^{EE}$  and  $C_l^{BB}$  in the sudden model have reionization bumps too low. This has been expected, since the half-gaussian fitting formula (34) is poor. The analytical  $C_l^{TT}$  and  $C_l^{TE}$  are close to the numerical ones on smaller scales  $l > 20$ , but have obvious departure from the numerical ones on very large scales  $l < 10$ . This implies that the approximation of temperature anisotropies  $\xi_l$  in Eq.(57) is poor for small multipoles  $l < 10$ . In the following we focus only on the two extended models and examine the impact of reionization through the analytical spectra  $C_l^{XX}$ .

## 5. Effects of Reionization

1. The most prominent modification due to the reionization is that it enhances the low- $l$  parts of the spectra, forming a reionization bump at  $l \sim 5$  for  $C_l^{EE}$  and  $C_l^{BB}$ , respectively. The position of this bump is a reflection of the horizon scale at reionization, whose corresponding angular scale,  $l \sim 5$ , is much larger than  $l \sim 100$  of the primary peaks at the photon decoupling. As pointed out earlier, the profiles of  $C_l^{XX}$  are determined by the profiles of RGWs at the decoupling and at the reionization as well. In particular, the reionization bumps are generated by  $\dot{h}(\eta_r)$ , and the primary peaks and troughs are due to  $h(\eta_d)$  and  $\dot{h}(\eta_d)$ . This correspondence is clearly demonstrated by Fig.9, in which the left panel plots  $C^{EE}$  and  $C^{BB}$ , as well as  $\dot{h}(\eta_d)$  and  $\dot{h}(\eta_r)$  in one graph, and the right panel plots  $C^{TT}$ , as well as  $h(\eta_d)$  and  $h(\eta_r)$  in one graph. This correspondence can be further explained by the following analysis. The respective projection factors,  $P_{Tl}$ ,  $P_{El}$ , and  $P_{Bl}$  as the integrands of  $C_l^{XX}$  are made up of the spherical Bessel's functions,  $j_l(x)$ , which is sharply peaked around  $x \simeq l$ . Subsequently the projection factors as functions of  $k$  are sharply peaked around

$$k(\eta_0 - \eta_d) \simeq k\eta_0 \simeq l, \quad (72)$$

$$k(\eta_0 - \eta_r) \simeq l, \quad (73)$$

Figure 9: The correspondence of the profiles of  $C_l^{XX}$  with that of RGWs at the decoupling and at the reionization.  $\dot{h}(\eta_r)$  is responsible for the bumps of  $C_l^{EE}$  and  $C_l^{BB}$  around  $l \sim 5$ , while  $\dot{h}(\eta_d)$  is responsible for the primary peaks and troughs for  $l \geq 100$ . For  $C_l^{TT}$ ,  $h(\eta_d)$  and  $h(\eta_r)$  have a similar slope at  $l \leq 10$ , and their superposition does not form a prominent bump of  $C_l^{TT}$ .

respectively. Consequently, the spectra as integrations over  $k$  will receive main contributions from the integration range  $k \sim l/\eta_0$  to the primary peaks and from  $k \sim l/(\eta_0 - \eta_r)$  to the bump, respectively [33]:

$$C_l^{EE}, C_l^{BB} \propto A_1^2 D_d^2(k) \left| \dot{h}(\eta_d) \right|_{k \sim l/\eta_0}^2 + A_2^2 D_r^2(k) \left| \dot{h}(\eta_r) \right|_{k \sim l/(\eta_0 - \eta_r)}^2, \quad (74)$$

$$C_l^{TT} \propto e^{-2\kappa_r} |h(\eta_d)|_{k \sim l/\eta_0}^2 + (1 - e^{-\kappa_r})^2 |h(\eta_r)|_{k \sim l/(\eta_0 - \eta_r)}^2, \quad (75)$$

$$C_l^{TE} \propto A_1 e^{-\kappa_r} D_d(k) h(\eta_d) \dot{h}(\eta_d)_{k \sim l/\eta_0} + A_2 (1 - e^{-\kappa_r}) D_r(k) h(\eta_r) \dot{h}(\eta_r)_{k \sim l/(\eta_0 - \eta_r)}. \quad (76)$$

According to Eq.(74), the locations of the primary peaks of  $C_l^{EE}$  and  $C_l^{BB}$  are mainly determined by the  $|\dot{h}(\eta_d)|^2$ -term, and those of the reionization bumps are determined by the  $|\dot{h}(\eta_r)|^2$ -term. However, the spectrum  $C_l^{TT}$  does not have a prominent bump around  $l \sim 5$ . This is because both  $|h(\eta_d)|^2$  and  $|h(\eta_r)|^2$  have a similar slope around there, and their superposition only enhances the spectral amplitude, not forming a bump. These are illustrated in Fig. 9.

2. The reionization bumps in the polarization spectra depend on the detailed reionization history.  $C_l^{EE}$  and  $C_l^{BB}$  for a fixed value of the optical depth  $\kappa_r = 0.084$  in the two extended models are shown in Fig. 10. The bumps in the  $\eta$ -linear model are located at a slightly larger angular scale (smaller  $l$ ) than that in the  $z$ -linear model. This is because we have assigned a greater  $\eta_r = 0.935$  in the  $\eta$ -linear model than that  $\eta_r = 0.855$  in the  $z$ -linear model, so its bump is located at a slightly smaller  $l \sim k(\eta_0 - \eta_r)$ . Notice also that the  $\eta$ -linear model produces higher bumps than the  $z$ -linear model. This is due the fact that the  $\eta$ -linear model has a greater width  $\Delta\eta_r = 0.286$  than that  $\Delta\eta_r = 0.855$  in the  $z$ -linear model. Thus we conclude that the location of bump is quite sensitive to the reionization time  $\eta_r$ , and the height of bump is sensitive to the width  $\Delta\eta_r$  of reionization process. This feature is helpful to probe  $\eta_r$  and  $\Delta\eta_r$  only if observational data on the bumps are accurate enough. However, when we let the two models to have the same set of parameters  $\eta_r$  and  $\Delta\eta_r$ , their reionization bumps predicted by our analytical formulation are very similar. The lesson is that the bump is an integrating result from the ionization fraction  $X_e(\eta)$ , and, in this regards, two different reionization histories via  $X_e(\eta)$  can lead to similar bumps, as long as they have similar  $V_r(\eta)$  [63, 64].

Figure 10:  $C_l^{EE}$  and  $C_l^{BB}$  in the extended reionization models. The two models yield different reionization bumps at  $l \sim 5$  since they are assigned with different values of  $\eta_r$  and  $\Delta\eta_r$ .

3. The overall profiles of CMB spectra are very sensitive to the optical depth  $\kappa_r$  of reionization. In particular,  $\kappa_r$  is strongly degenerate with the normalization of the amplitude of primordial fluctuations, and this fact has been one of main difficulties to probe the details of reionization process [9, 10, 11, 12, 13, 65, 66, 67]. It should be emphasized that the reionization does not change the primordial amplitude  $A$  of RGWs in Eq.(6), which is implicitly contained in  $h(\eta)$  and  $\dot{h}(\eta)$ . The impact of  $\kappa_r$  is through the coefficients  $A_1(\kappa_r)$  and  $A_2(\kappa_r)$  in  $\beta_l$  in Eq.(50), as well as the coefficients  $e^{-\kappa_r}$  and  $(1 - e^{-\kappa_r})$  in  $\alpha_l$  in Eq.(58). The main features of the  $\kappa_r - A$  degeneracy are clearly revealed by the analytical estimations in Eqs.(74), (75), and (76).

For instance, look at Eq.(74) for  $C_l^{EE}$  and  $C_l^{BB}$ . A larger  $\kappa_r$  gives smaller  $A_1$  and larger  $A_2$ , leading to lower primary peaks and higher bumps of  $C_l^{EE}$  and  $C_l^{BB}$ , as illustrated in Fig. 11. But, this lowering of primary peaks can be compensated by a choice of a higher amplitude normalization  $A$ , which enhances the amplitude of  $\dot{h}(\eta_d)$ , resulting in the unchanged term  $A_1^2(\kappa_r)|\dot{h}(\eta_d)|^2$ , so that the primary peaks remain the same. This is the  $\kappa_r - A$  degeneracy. Similar degeneracy in  $C_l^{TT}$  and  $C_l^{TE}$  are also understood by Eq.(75) and Eq.(76).

The  $\kappa_r - A$  degeneracy can be broken. Again, take  $C_l^{EE}$  and  $C_l^{BB}$  as an example. While a larger  $\kappa_r$  and a higher  $A$  can yield the unchanged primary peaks, the reionization bumps get doubly enhanced, since the bump term  $A_2^2(\kappa_r)|\dot{h}(\eta_r)|^2$  in Eq.(74) gets doubly enhanced. This suggests a possible way to break the degeneracy. Eq.(74) tells that the relative height of the primary peaks and the bump is given by

$$\frac{\text{primary peak amplitude}}{\text{bump amplitude}} \propto \frac{A_1^2(\kappa_r) |\dot{h}(\eta_d)|^2}{A_2^2(\kappa_r) |\dot{h}(\eta_r)|^2}. \quad (77)$$

For any given RGWs, the ratio  $|\dot{h}(\eta_d)| / |\dot{h}(\eta_r)|$  is independent of  $A$  and completely determined, so one has

$$\frac{\text{primary peak amplitude}}{\text{bump amplitude}} \propto \left( \frac{A_1(\kappa_r)}{A_2(\kappa_r)} \right)^2. \quad (78)$$

This ratio only depends on the value of  $\kappa_r$  and is not sensitive to the details of a reionization model. Therefore, using this ratio of heights, one can infer the value of  $\kappa_r$  from the observational data of  $C_l^{EE}$  and  $C_l^{BB}$ , thus breaking the degeneracy.

4. The primordial fluctuation spectral index  $\beta_{inf}$  introduced in Eq.(6) is a very important parameter for inflationary models. Given a normalization  $A$  of the RGWs amplitude, a large

Figure 11: The  $\kappa_r - A$  degeneracy. A larger value of  $\kappa_r$  enhances the bumps at  $l \sim 5$  and, at the same time, reduces the primary peaks of  $C_l^{EE}$  and  $C_l^{BB}$ . The plot is made for the  $z$ -linear model. This degeneracy behavior also exists in the  $\eta$ -linear model.

Figure 12: The  $\kappa_r - \beta_{inf}$  degeneracy. Although the bumps and the 1<sup>st</sup> primary peaks are degenerate, the 2<sup>nd</sup> and 3<sup>rd</sup> primary peaks show clear departure. The plot is made for the  $z$ -linear model.

$\beta_{inf}$  tilts the spectrum  $h(\nu, \eta_i)$ , in such a way that RWGs is more strongly enhanced on smaller scales [39, 38]. The RGWs-generated spectra  $C_l^{XX}$  are subsequently tilted in the same way [33, 34]. Therefore, a larger  $\beta_{inf}$  brings about a similar effect on  $C_l^{XX}$  as a smaller  $\kappa_r$  does, leading to certain bias in determining  $\kappa_r$  [67, 68, 69, 70, 71]. Take  $C_l^{EE}$  and  $C_l^{BB}$  as an example, for which the effect is more prominent. Fig. 12 shows that, for the  $z$ -linear model, the case ( $\beta_{inf} = -2.02, \kappa_r = 0.106$ ) and the case ( $\beta_{inf} = -2.10, \kappa_r = 0.084$ ) yield almost overlapping curves of the bump and the 1<sup>st</sup> primary peak as well.

The  $\kappa_r - \beta_{inf}$  degeneracy can also be understood by the analytical estimation in Eq.(74). While a large  $\beta_{inf}$  enhances  $|\dot{h}(\eta_d)|^2$  on small scales, a large  $\kappa_r$  suppresses  $A_1(\kappa_r)$ , resulting in an unchanged combination  $A_1(\kappa_r)^2 |\dot{h}(\eta_d)|^2$  for the primary peaks. But this degeneracy is clearly broken from the 2<sup>nd</sup> primary peak on. This is because the  $\kappa_r$ -induced change in  $A_1(\kappa_r)$  is scale-independent, whereas the  $\beta_{inf}$ -induced change in  $|h(\eta_d)|$  depends on the scale. Therefore, one expects that data of the smaller scale  $C_l^{EE}$  and  $C_l^{BB}$  will be helpful in breaking the  $\kappa_r - \beta_{inf}$  degeneracy. Note also that the principal component method developed in Ref.[71] can protect the bias of  $\kappa_r$  caused by  $\beta_{inf}$ .

5. Although the magnetic type of polarization  $C_l^{BB}$  is thought to be a “smoking gun” of detection of RGWs, its detection is not done yet, which may be accomplished by a future CMBpol experiment [72]. The 5-year WMAP [5, 6] has given the observed cross-spectrum  $C_l^{TE}$ , which is negative (anti-correlation) in a range  $l \sim (50, 220)$ . Yet this observed  $C_l^{TE}$  is a superposition of contributions by both scalar perturbations and RGWs. In order to extract the traces of RGWs out of  $C_l^{TE}$ , one still needs to disentangle the contribution by RGWs from the total. In the so-called zero-multipole method [25, 45, 73, 74], one examines the impact of the tensor/scalar ratio  $r$  upon the zero multipole  $l_0$  around  $\sim 50$ , where  $C_l^{TE}$  first crosses the value 0 and turns negative. However, there are other factors that can influence the value of  $l_0$ . The variation of  $l_0$  caused by NFS has been estimated to be small  $\Delta l \leq 4$  [34]. Here the reionization is another important factor that brings about a change of  $l_0$ , as is shown in Fig. 13 for the extended reionization

Figure 13: Reionization shifts  $C_l^{TE}$  to smaller angular scales by  $\Delta l \sim 20$  around the region  $l \sim 50$ . For illustration  $\kappa_r = 0.084$  and  $r = 0.37$  have been taken.

Figure 14:  $C_l^{TE}$  by the baryon isocurvature mode is positive around  $l \sim 100$ , whereas that by RGWs is negative there. Only when a very small  $r = 0.001$  is taken, is the amplitude of  $C_l^{TE}$  by the isocurvature modes comparable to that by RGWs. The  $C_l^{TE}$  by isocurvature mode is the numerical result generated using CAMB [16] with  $\alpha_{-1} = 0.015$ . .

models with  $\kappa_r = 0.084$ . Around the relevant region of  $l \sim 50$ , the reionization shifts the curve of  $C_l^{TE}$  to smaller angular scales by an amount of  $\Delta l \sim 20$ , in comparison with the non-reionized  $C_l^{TE}$ . This amount is much larger than that caused by NFS. Moreover, the shift  $\Delta l$  increases with the optical depth  $\kappa_r$ . This significant effect has to be incorporated into the zero multipole analysis before one can make an extraction of RGWs from the total  $C_l^{TE}$ .

In this procedure, besides disentangling the adiabatic (constant entropy) modes that are dominant in the scalar perturbations, one need consider the isocurvature modes possibly existing in the cosmological plasma [75], which can contribute to  $C_l^{XX}$  [76]. In particular, the isocurvature modes contribute positively (correlation) to the cross spectrum  $C_l^{TE}$  in the range around  $l \sim 100$ , in contrast to the adiabatic modes, which contribute negatively (anti-correlation) there. The observed  $C_l^{TE}$  from WMAP has shown the anti-correlation, and a very stringent constraint has been found on the isocurvature contribution with the isocurvature/adiabatic ratio  $\alpha_{-1} < 0.015$  at 95%CL [5]. It is interesting to compare the contributions from RGWs and isocurvature perturbations to  $C_l^{TE}$ . The comparison is very sensitive to the ratio  $\alpha_{-1}$  and the tensor/scalar ratio  $r$ . Taking the upper limit  $\alpha_{-1} = 0.015$  constrained from WMAP-5, and using the CAMB Online Tool [16] results for isocurvature modes of the plasma components of baryon, CDM, and neutrino, one finds that when  $r = 0.37$  is taken, the amplitude of  $C_l^{TE}$  generated by RGWs is about two orders greater than that of the isocurvature modes. So in this case the isocurvature can be neglected. Only when a much smaller ratio  $r = 0.001$  is taken, is the contribution by the isocurvature modes comparable to that by RGWs. This is demonstrated with  $r = 0.001$  and  $r = 0.01$  in Fig.14, in which the numerical  $C_l^{TE}$  contributed by the baryon isocurvature perturbation has been produced from CAMB [16] with  $\alpha_{-1} = 0.015$ .

6. So far our analytic formulation for reionization can only distinguish two different extended models by comparing their  $\kappa_r$ ,  $\eta_r$ , and  $\Delta\eta_r$ . The damping factor  $D_r(k)$  in Eq.(46) as a fitting formula could be used to specify other fine details of two reionization models. Obviously, with a fixed  $b$ , a larger  $c$  leads to lower bumps of  $C_l^{EE}$  and  $C_l^{BB}$ , as shown in Fig. 15 for the  $z$ -linear



Figure 15: The damping factor  $D_r(k)$  in Eq.(46) depends on the parameters  $c$  and  $b$ . The plot is made for the  $z$ -linear model.

model. On the other hand, with a fixed  $c$ , a larger  $b$  will yield a slightly higher reionization bumps, while leaving the primary peaks almost intact. This property can be inferred as the following. For the reionization bump around  $l \sim 5$ , the contribution is mainly from  $k \sim l/(\eta_0 - \eta_r)$  according to Eq.(73), so  $D_r(k) \propto e^{-c(\Delta\eta_r l/(\eta_0 - \eta_r))^b}$ . In the reionization models considered in this paper, the combination  $\Delta\eta_r l/(\eta_0 - \eta_r) \sim 0.5 < 1$ , so a larger parameter  $b$  leads to a larger  $D_r$  and higher bumps. For the primary peaks with  $l \geq 100$ ,  $D_r(k)$  is so small that the term  $A_2^2 D_r^2(k) \left| \dot{h}(\eta_r) \right|_{k \sim l/(\eta_0 - \eta_r)}^2$  to  $C_l^{EE}$  and  $C_l^{BB}$  is practically negligible, leaving the primary peaks intact under a variation of  $b$  in  $D_r(k)$ .

## 6. Summary

We have presented the approximate, analytical formulation of the reionized CMB spectra  $C_l^{XX}$  generated by RGWs. Even though its approximate nature implies its application as a complement to the numerical codes, it does improve our understanding CMB and efficiently promote the analysis of various effects that reionization brings upon  $C_l^{XX}$ .

The reionization around  $z \sim 11$  is studied by three simple homogeneous models, i.e., a sudden reionization, two extended reionizations with ionization fraction  $X_e(\eta) \propto \eta$  and  $X_e(\eta) \propto z$ . The key parameter is  $\kappa_r$ , the optical depth from the present back up to the start of reionization. Given a value of  $\kappa_r$  in each model, the visibility function  $V_r(\eta)$  follows, which is approximately fitted by Gaussian type functions. This procedure is similar to the treatment of decoupling in our previous study.

Then the time integrations for polarization mode  $\beta_l$  and temperature anisotropies mode  $\alpha_l$  are carried out approximately, and the resulting analytic expressions consist of contributions by RWGs  $h(\eta_d)$  and  $\dot{h}(\eta_d)$  at the decoupling, and by  $h(\eta_r)$  and  $\dot{h}(\eta_r)$  at the reionization as well. It is found that, while  $h(\eta_d)$  and  $\dot{h}(\eta_d)$  generate the primary peaks at  $l \geq 100$ ,  $\dot{h}(\eta_r)$  produces bumps for  $C_l^{EE}$  and  $C_l^{BB}$  at  $l \sim 5$ , and  $h(\eta_r)$  enhances  $C_l^{TT}$  and  $C_l^{TE}$  there. The analytic  $C_l^{XX}$  qualitatively agree with those by the numerical computing, such as CAMB.

As a merit of our analytic approach, the dependence of  $C_l^{XX}$  upon the optical depth  $\kappa_r$  are explicitly given, in terms of the coefficients  $a_1(\kappa_r)$  and  $a_2(\kappa_r)$  for the polarization  $\beta_l(\eta_0)$ , and of the coefficients  $e^{-\kappa_r}$  and  $(1 - e^{-\kappa_r})$  for the temperature anisotropies  $\alpha_l(\eta_0)$ . It is found that  $a_1(\kappa_r)$  and  $a_2(\kappa_r)$  vary with  $\kappa_r$  more quickly than  $e^{-\kappa_r}$  and  $(1 - e^{-\kappa_r})$ , respectively. Therefore, the polarization  $\beta_l$  is more sensitive to  $\kappa_r$  than the temperature anisotropies  $\alpha_l$  does. A larger

$\kappa_r$  gives higher  $a_2(\kappa_r)$  and lower  $a_1(\kappa_r)$ , i.e., yielding higher bumps and lower primary peaks in  $C_l^{EE}$  and  $C_l^{BB}$ . Thus there is a degeneracy of  $\kappa_r$  with the normalization of the initial amplitude  $A$  of RGWs. Besides,  $\kappa_r$  also has a weak degeneracy with the spectral index  $\beta_{inf}$  of RGWs since a larger  $\beta_{inf}$  enhances the primary peaks on small scales. The analytical  $C_l^{EE}$  and  $C_l^{BB}$  also suggest possible ways to break these two kinds of degeneracies.

Besides  $\kappa_r$ , our formulation also demonstrates the effects of the reionization time  $\eta_r$  and the reionization duration  $\Delta\eta_r$ . For a fixed  $\kappa_r$ , the height of bump is proportional to  $\Delta\eta_r$ , and the location  $l$  of bump depends on  $\eta_r$  in such a way  $l \sim k(\eta_0 - \eta_r)$  that a later reionization (larger  $\eta_r$ ) yields a bump at larger angular scales (smaller  $l$ ).

Given a fixed set of parameters  $\kappa_r$ ,  $\eta_r$ , and  $\Delta\eta_r$ , the  $\eta$ -linear and  $z$ -linear models yield similar bumps in  $C_l^{EE}$  and  $C_l^{BB}$ . Thus our analytical formulation is unable to rediscover the reionization history  $X_e(\eta)$  from  $C_l^{XX}$ .

These analytical results tell that studies of reionization by means of CMB temperature anisotropies and polarization not only requires sufficient observational data, but also need detailed studies of the reionization process itself and more realistic modeling.

The reionization process also significantly affects the possible detections of RGWs via the observations of  $C_l^{XX}$ . In particular, it is found that the reionization causes a shift of the zero multipole  $l_0$  of the cross spectrum  $C_l^{TE}$  by a substantial amount  $\Delta l \sim 20$ , which is also  $\kappa_r$ -dependent. The effect of reionization need be properly included, before one can apply the zero multipole method to extract the traces of RGWs from the observed  $C_l^{TE}$ .

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